Determination of intersections between Cartesian planes and raster digital elevation models

Algorithm Theoretical Basis Document

for *qgSurf* plugin

Start editing: 2018-11-25
Vers. 2.0 - last editing: 2019-01-06
Completed: No

The intersection between a geological plane and a topographic surface is a subject of interest in geological field mapping and structural geology.
In a GIS, the topographic surface is generally represented as a grid. GDAL, the most diffuse open-source library for handling rasters, describes and manages a grid via parameters such as the cell sizes, the top-right cell corner coordinates and also the grid rotations.
Generally available topographic grids do not present rotations, but this possibility cannot be ruled out in a few cases.
Prior to the 2.0 version, the Python implementations in gsf and qgSurf algorithms assumed no grid rotations for determining the plane-grid intersections. With the 2.0 version, a new algorithm, inspired to the previous one, permit to process also grid with built-in rotations.

1. The GDAL geotransform

To describe the geographical properties of a raster, GDAL uses the concept of "geotransform".
In essence, the GDAL geotransform is just a matrix that allows to derive the geographic coordinates of a point given its pixel coordinates.
Mathematically, it can be expressed using an augmented matrix:

(1)
$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} A & | & \vec{b} \\ 0..0 & | & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

so that:

(2)
$$y' = A\vec{x} + \vec{b}$$

In eq. (1) we can substitute the geotransform parameters:

(3)
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} gt_1 & gt_2 & gt_0 \\ gt_4 & gt_5 & gt_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The linear equations for the transformed coordinates are:

(4)
$$\begin{aligned} x' &= gt_1 x + gt_2 y + gt_0 \\ y' &= gt_4 x + gt_5 y + gt_3 \\ 1 &= 0 + 0 + 1 \end{aligned}$$

From the equation in (4), it can be easily observed that the gt0 and gt3 represent grid offsets in the x - and y- directions respectively, gt1 and gt5 are scaling factors for x- and y- directions, while gt2 and gt4 represent rotations/skewing.

In the general case, an orthogonal grid can be transformed by a GDAL geotransform into a grid where x and y axes are no longer perpendicular: an example of a skewed and rotated grid is illustrated in Fig. 1.

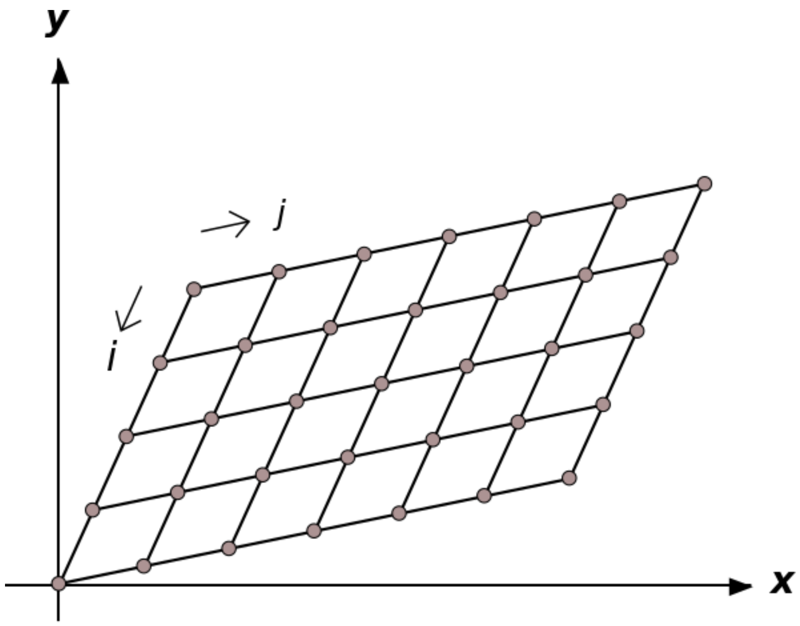

Figure 1: Example of an orthogonal grid transformed by an affine transformation.

2. Theoretical determination of plane-DEM intersections

To determine the plane-DEM intersections, we transform the geometrical problem from 3D to 2D, in order to simply it. We consider the geometric traces of the plane and of the topographic surface with. two sets of parallel vertical planes, the first set oriented parallel to the final j- grid axes orientations, the second set parallel to the final i- grid axis orientation (see Fig. 1). Each plane of the set contains a grid point row (for the j-parallel planes) or column (for the i-parallel planes). The points correspond to the final grid cell centers.

Now we turn to a single vertical plane, for example in the j-direction (see Fig. 2, that correspond to a vertical section).

We have equi-spaced final cell centers along the plane, where the spacing do not necessarily corresponds to the original geotransform cell sizes, given the a general case geotransform can distort (skew) a grid. This spacing is however constant in each set of vertical planes and can be easily calculated as the distance between the first and second geotransformed cell centers. When there is no grid skewing and just a rigid-body rotation (o no rotation at all), the spacing will be equal to the original grid cell sizes.

Within a single vertical plane we consider its intersections with the DEM and the plane surfaces, at each geo-transformed cell centers.

The point intersection with the DEM corresponds trivially to the DEM height for that cell, while the plane intersections can be easily calculated given the plane equation and the considered cell center point coordinates.

Knowing for each cell center along a vertical plane the DEM and plane height, we can determine the plane-DEM intersection between two consecutive cell centers, as described below.

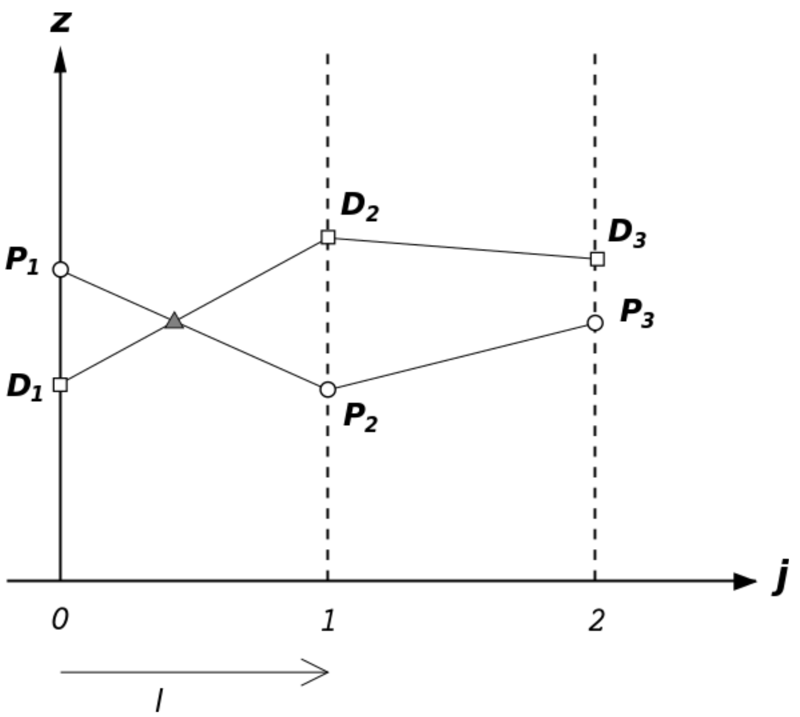

Figure 2: Determination of intersecting plane (P points) and DEM (D points) along the j direction.

As can be seen in Fig. 2, we have a valid plane-DEM intersection point between two consecutive cell centers (for instance j=0 and j=1 in Fig. 2), when the relative z positions of the plane and the DEM traces switch between the two considered cell centers. On the other hand, when the plane line is always higher (or lower) than a DEM line, there is no intersection (e.g., j=1 and j=2 in Fig. 2). Obviously, when at a cell center line and DEM points coincide, it correspond to an intersection point.

The height equations for the DEM and the plane are:

$$z_d = m_d x' + q_d$$
$$z_p = m_p x' + q_p$$

where q_p is the plane elevation at point (x, y), q_d is the grid z value, m_d is the angular coefficient of the DEM trace (for the given direction) and m_p is the angular coefficient of the plane trace in the considered direction.

At an intersection point we have:

$$m_d x' + q_d = m_p x' + q_p \Rightarrow$$
$$(m_d - m_p) x' = q_p - q_d \Rightarrow$$
$$x' = \frac{q_p - q_d}{m_d - m_p}$$

where x' is the distance of the intersection point from the left cell center.

The array coordinate of the intersection point is therefore equal to:

$$\frac{x'}{cellsize'} = i'[0 \rightarrow 1]$$

where *cellsize'* is equal to the geotransformed cell spacing in the considered direction and *i'* is the array coordinate in the considered direction, that can then be transformed into geographic coordinates by applying the geotransform, thus solving the investigated problem.

3. Practical determination of plane-DEM intersections: geological plane pre-processings

In GIS, the projection used by the GIS application, e.g., QGIS, may differ from the one of the DEM chosen for the intersections determination.

This fact has two practical implications for the processing of the geological plane orientation:

1) in the projection used by the current project, the top direction (y-axis direction) can be not parallel to the geographic North;

2) in the case of project-DEM projections difference, there can be a variation of the orientations and length of corresponding lines between the two projective systems.

We therefore need two preliminary corrections to the geological plane attitude, the former related to the geographic North disorientation angle with regards to the project CRS y-axis, the latter to the change from project- to DEM-CRS and its impact on the lenght and orientation of a segment parallel to the geological plane dip-direction.

3.1 Correction for disorientation of geographic North in current project CRS

When we consider the orientation of geological planes, we are referring to it as the plane dip direction azimuth with respect to the geographic North. Since the algorithmic calculation refers to the Cartesian y-direction (up-direction) in the current (*i.e.*, project-defined) projection, we have to correct the user-provided orientation for the horizontal angular disorientation between the geographic North and the y-axis.

Morevover, in the general case, this angular disorientation cannot be assumed to be constant in the whole of the considered area (ie., the DEM extent).

In order to simplify the correction calculation, just one location is considered, the user-defined source point of the geological plane. From the practical point of view, this is the location of most interest/significance for the user, so it is also important to have the highest intersection correctness in its neighbors.

Moreover for the considered cases the obtained correction is also minimal (less than 1°), so its variation in the full DEM extent are to be considered practically irrelevant, considering also all of the other possible uncertainties in the concrete scientific analysis.

This angular disorientation is calculated in the following way:

1. the user-defined source point is converted from the original project CRS to a latitude-longitude framework;

2. in the latitude-longitude framework, a second dummy point is created just North of the source point, at a distance of around 90 meters (3 arc-seconds);

3. the source and the dummy North points are projected in the project CRS;

4. in the project CRS, the angle between the projected North-oriented segment and the y-axis is calculated;

5. the azimuth correction (angle between geographic North and y-axis, measured clockwise) is added to the user-provided geological plane dip-direction.

3.2 Correction for the line transformation between project CRS and DEM CRS

Since the calculations to derive the intersection points are implemented, for algorithmic simplicity, in the DEM CRS space, we have to consider the impact of CRS change, i.e, to DEM CRS from project CRS, on the geological plane attitude.

A segment can be transformed in two way: its spatial orientation can change, as well as its length.

When the reference segment (parallel to the dip direction of the geological plane) changes orientation, it requires a correction of the azimuth of the dip direction, similarly to the previously described azimuth correction for the geographic North disorientation in the project CRS.

When there is a change in the horizontal length of the reference segment, it impacts the dip angle in the DEM CRS, since the depth is not distorted in a proportional way in the reprojection (fig. DIP-ANGLE-CRS-CHANGE-IMPACT).

The correction is derived in a way similar to the previously described corrections.

Using the methods for vector manipulation made available in the *pygsf* library, a segment, parallel to the (North-corrected) geological plane dip-direction and with a starting length of 100 distance units (e.g., meters) is created in the project CRS space. It is projected in the DEM CRS space and its new orientation and lengths are extracted. These two information are used to update the geological plane dip direction and angle (? ALTRA FIG. ESPLICATIVA).